VIII Non-characterizing Surgeries

here are two natural questions:

I) For a fixed n, are there infinitely many knots $K_1, K_2, \dots,$ such that $S^3_{K_i}(n) \cong S^3_{K_j}(n)$?

II) For a fixed n, are there infinitely many knots $K_1, K_2, \dots$ such that $X_{K_i}(n) \cong X_{K_j}(n)$?

recall this is the 4-manifold obtained from B^4 by attaching a 2-handle to K_i w/ framing n

call this the <u>n-trace</u> of the knot

Clearly Yes to II) $\Rightarrow$ Yes to I)

(since $\partial X_{K_i}(n) = S^3_{K_i}(n)$)

A. Annulus Twists

We will see the answer to both questions is Yes

We start with a construction called <u>annulus twists</u>

lemma 1: let $A \subset M^3$ be an embedded annulus with boundary $K_1 \cup K_2$

Suppose $\mathcal{F}$ is the framing on K_i coming from A

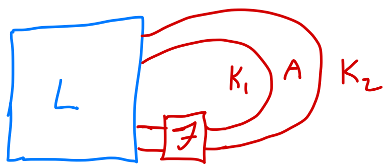
Then $M_{K_1 \cup K_2}(\mathcal{F} + 1/n, \mathcal{F} - 1/n)$ is diffeomorphic to M

Proof: for $n = 1$, note $M_{K_1 \cup K_2}(\mathcal{F} + 1, \mathcal{F} - 1)$ is the same manifold as the one obtained by cutting M along A and regluing by a negative Dehn twist on K_1 and positive Dehn twist along K_2 (see Lemma I.6) but of course this diffeomorphism of A is isotopic to the identity and so yields M

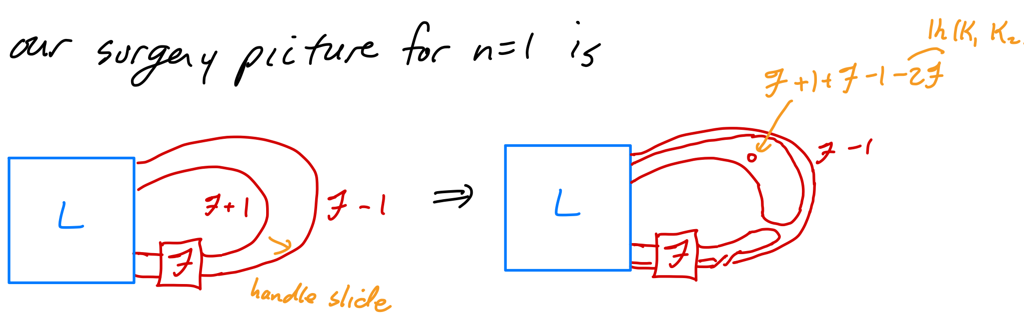
for large n note that if you take n copies of K_i pushed off with framing $\mathcal{F}$ then ± 1 Dehn surgery on all of them is the same as $\mathcal{F} \pm 1/n$ Dehn surgery on K_i

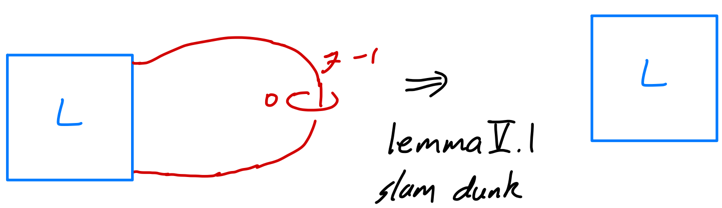
exercise: Check this

Second proof: there is some Dehn surgery presentation for M and in there we see A



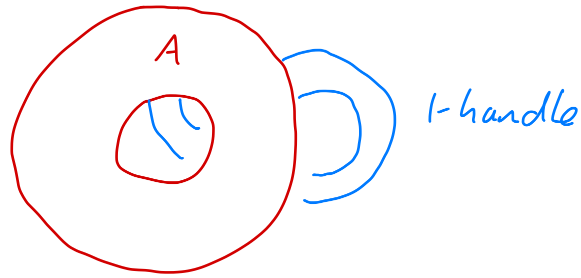
so our surgery picture for $n=1$ is





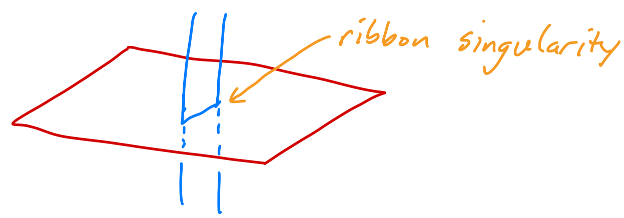
exercise: prove the general n case

let $\Sigma = A \cup$ 1-handle (annulus: A) so that Σ has one boundary component



consider an immersion $\phi : \Sigma \to M$ such that

- $\phi|_A$ is an embedding
- $\phi|_{\text{1-handle}} \cap \text{int } A$ are ribbon singularities



this is called an annulus presentation or band presentation for the knot $K = \phi(\partial \Sigma)$

example:

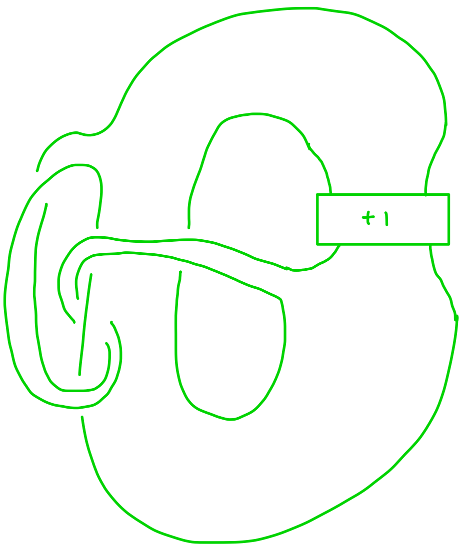
let A' be a subannulus of A st. $\phi(\text{int } A')$ contains all the ribbon singularities and set $\partial A' = K_1 \cup K_2$ with framing $\mathcal{F}$ comming from A'

from the lemma above $M_{K_1 \cup K_2}(\mathcal{F} + \frac{1}{n}, \mathcal{F} - \frac{1}{n}) \cong M$

but what happens to K ?

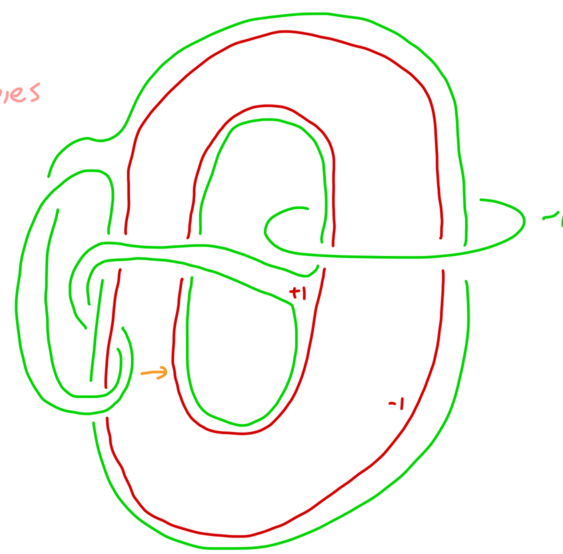
before you cancel the surgeries on K_1 and K_2 in the proof above slide K over K_1 or K_2 at the end points of the ribbon singularity

example:

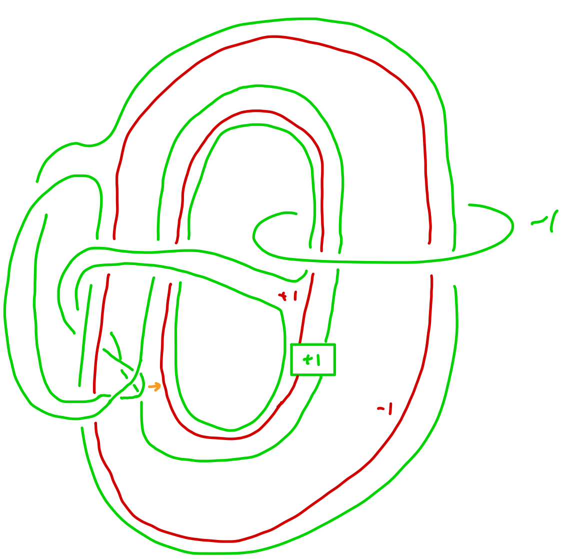


to make it easier to see the isotopies

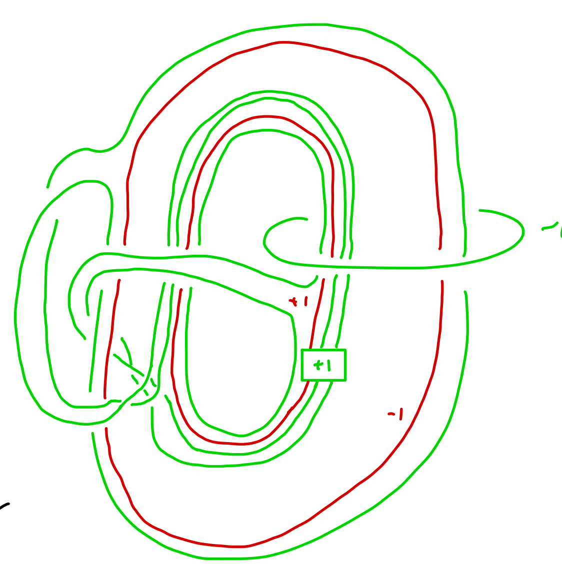
blow up



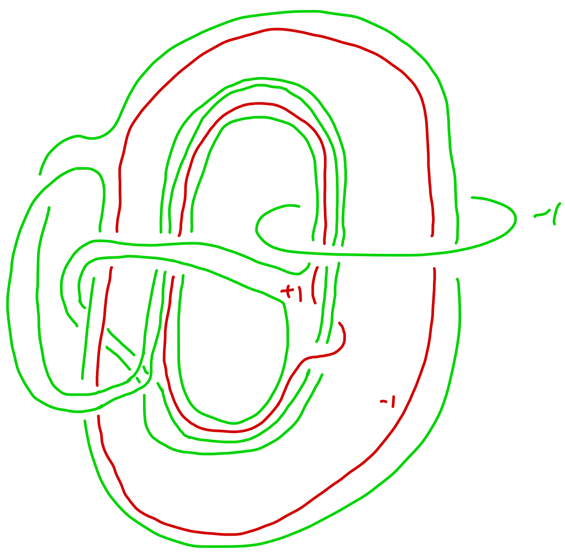
handle slide and isotopy



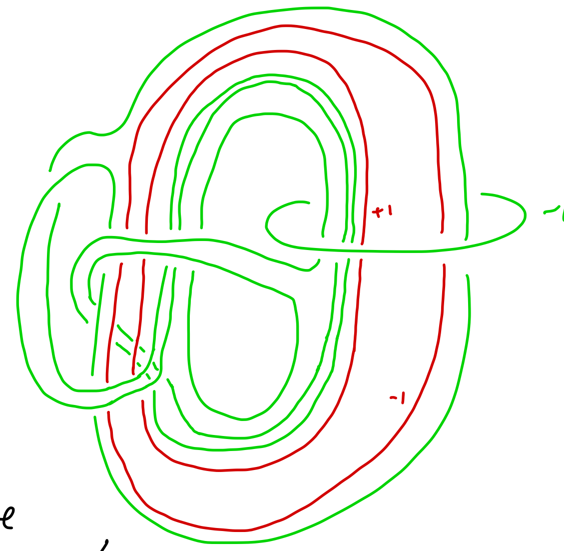
handle slide



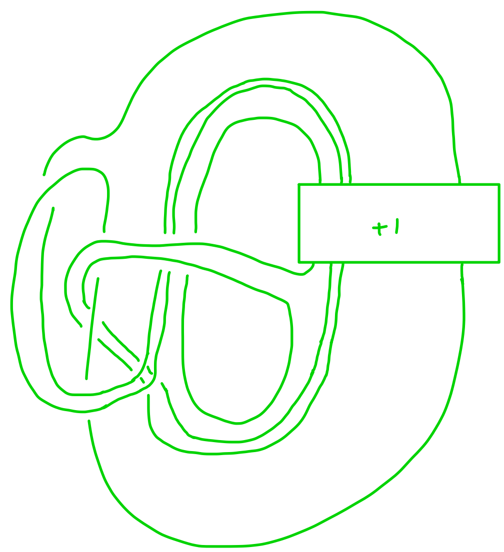
isotopy



$\xrightarrow{\text{isotopy}}$

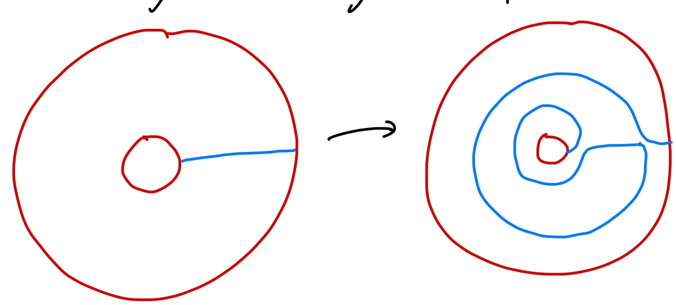


remove cancelling pair and blow down



exercise: understand this from the perspective of the first proof of lemma 1

Hint: we cut M along A and reglues by



this is clearly isotopic to the identity

use isotopy to give explicite diffeom.
from $M_{K_1 \cup K_2}(\mathcal{F}+1/n, \mathcal{F}-\frac{1}{n})$ to M
see where K goes!

define $A^n(K)$ = image of K under diffeomorphism

$$M_{K_1 \cup K_2}(\mathcal{F}+\tfrac{1}{n}, \mathcal{F}-\tfrac{1}{n}) \cong M$$

we say $A^n(K)$ is obtained from K by an <u>annulus twist</u>

let $\mathcal{F}'$ be the framing on K (and $A^n(K)$) induced by $\phi(\Sigma)$

<u>exercise</u>: Compute $\mathcal{F}'$ if $M = S^3$

<u>Th^m 2 (Osoinach 2006)</u>:

$$\boxed{M_K(\mathcal{F}') \cong M_{A^n(K)}(\mathcal{F}') \text{ for all } n}$$

<u>Proof</u>: consider $\Sigma' = \Sigma - A'$ (note: pair-of-pants)
note that in $M_K(\mathcal{F}')$ you glued a meridional disk
to $\Sigma' \subset M - \text{nbhd}(K)$ along longitude for nbhd(K)
so $\Sigma' \cup$ meridional disk is an annulus $\bar{A}$ in $M_K(\mathcal{F}')$
note $K_1 \cup K_2 = \partial \bar{A}$ and the framing $\mathcal{F}$ on K_1, K_2 from
A' = framing on $K_1 \cup K_2$ from $\bar{A}$
$\therefore$ by lemma 1, $\mathcal{F}+1/n, \mathcal{F}-1/n$ surgery on $K_1 \cup K_2$
in $M_K(\mathcal{F}')$ yields $M_K(\mathcal{F}')$

but I could do surgery on the $K_1 \cup K_2$ first to get $A^n(K)$ in M and then $\mathcal{F}'$ surgery on $A^n(K)$ to get $M_K(\mathcal{F}')$ $\blacksquare$

Cor 3:

$\boxed{\text{If } K \text{ is as in example above, then } A^n(K) \text{ different for each } n, \text{ so } \exists\, \infty\text{'ly many knots in } S^3 \text{ on which 0-surgery yields the same 3-manifold}}$

Proof: $K \cup K_1 \cup K_2$ (∂ of pair-of-pants) is hyperbolic (use SnapPy a computer program good at dealing with hyperbolic manifolds)

thus by Thurston's hyperbolic Dehn surgery theorem for large n, $A^n(K)$ is also hyperbolic and as $n \to \infty$ its volume increases to that of $K \cup K_1 \cup K_2$ so they are all different! $\blacksquare$

Remark: If you know about other, easier, knot invariants you might try to show the Alexander polynomials or signatures of the $A^n(K)$ are different but since $S^3_{A^n(K)}(0) \cong S^3_K(0)$ one can check that their Alexander modules are the same (recall these are determined by $\pi_1(S^3 - K)$ how does this relate to $\pi_1(S^3_K(0))$?). So the Alexander polynomials and signatures are the same.

given an annulus presentation $(A, \text{1-handle})$ of a knot K

we say it is special if

1) $A =$ a ± 1 twisted band about an unknot bounding disk D, and
2) the 1-handle is disjoint from D.

note: our example above is special

Thm 4 (Abe-Jong-Omae-Takeuchi, 2013):

> If K has a special annulus presentation then
> $$X_K(0) \cong X_{A^n(K)}(0)$$
> for all n

the proof relies on a result of Akbulut

lemma 5 (Akbulut, 1977):

> let K, K' be knots in ∂B^4 with a diffeomorphism
> $$g : \partial X_K(n) \to \partial X_{K'}(n)$$
> and let μ be a meridian of K. Suppose
>
> (1) if μ is 0-framed, then $g(\mu)$ is a 0-framed unknot in the Kirby diagram representing $X_{K'}(n)$ and
>
> (2) the Kirby diagram $X_{K'}(n) \cup h'$ represents B^4, where h' is the 1-handle represented by a dotted $g(\mu)$
>
> then g extends to a diffeomorphism $X_K(n) \to X_{K'}(n)$

Proof: note: μ is the boundary of the co-core of the 2-handle in $X_K(n)$

thus it bounds a disk D, the co-core of the handle

recall removing a nbhd of the co-core is the same as removing the handle

so $X_K(n) \setminus \nu(D) \cong B^4$

by hypothesis $g(\mu)$ bounds a disk D' in $X_{K'}(n)$ and $X_{K'}(n) \setminus \nu(D) \cong B^4$

recall, $\nu(D) = D \times D^2$ and this framing induces the 0-framing on $\partial D^2 \subset \partial X_K(n)$

similarly for $\nu(D')$

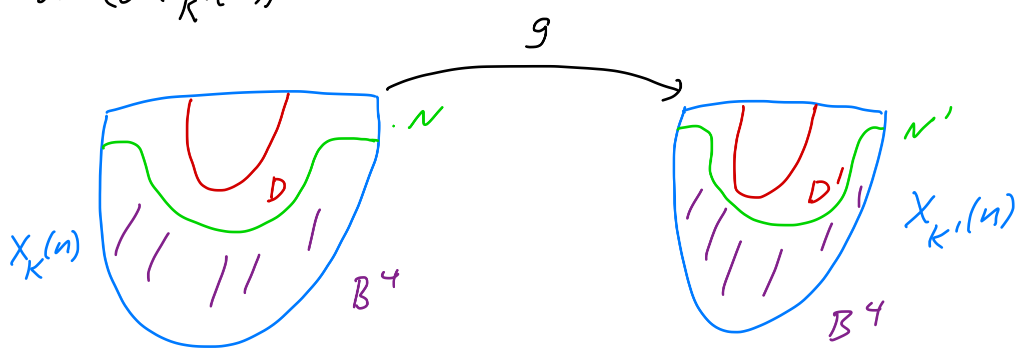
so a

$$\text{nbhd}(\partial X_K(n) \cup D) = [(\partial X_K(n)) \times [-1,0]] \cup \text{2-handle attached to } \mu \text{ w/ framing } 0$$

and

$$\text{nbhd}(\partial X_{K'}(n) \cup D') = [(\partial X_{K'}(n)) \times [-1,0]] \cup \text{2-handle attached to } g(\mu) \text{ w/ framing } 0$$

thus g can be extended to a diffeomorphism G from a neighborhood N of $(\partial X_K(n)) \cup D$ to a neighborhood N' of $(\partial X_{K'}(n)) \cup D'$



now $\overline{X_K(n) - N} \cong B^4$ and $\overline{X_{K'}(n) - N'} \cong B^4$

and $G|_{\partial(X_K(n)-N)} : \partial B^4 \to \partial B^4$

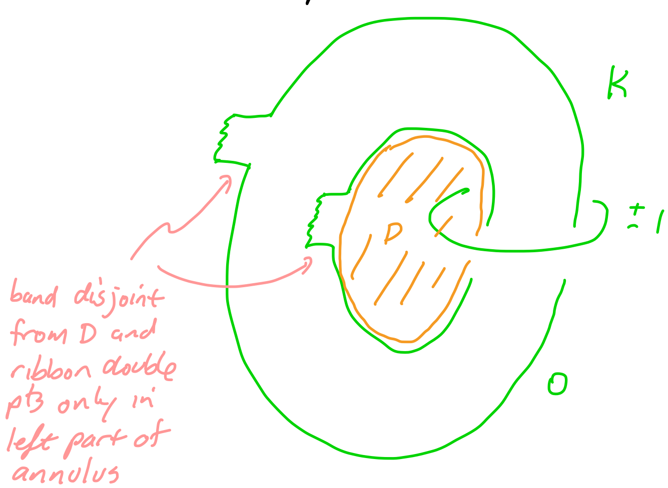
Fact (Cerf 1968):

$\boxed{\text{any diffeomorphism of } S^3 \text{ extends to a diffeomorphism of } B^4}$

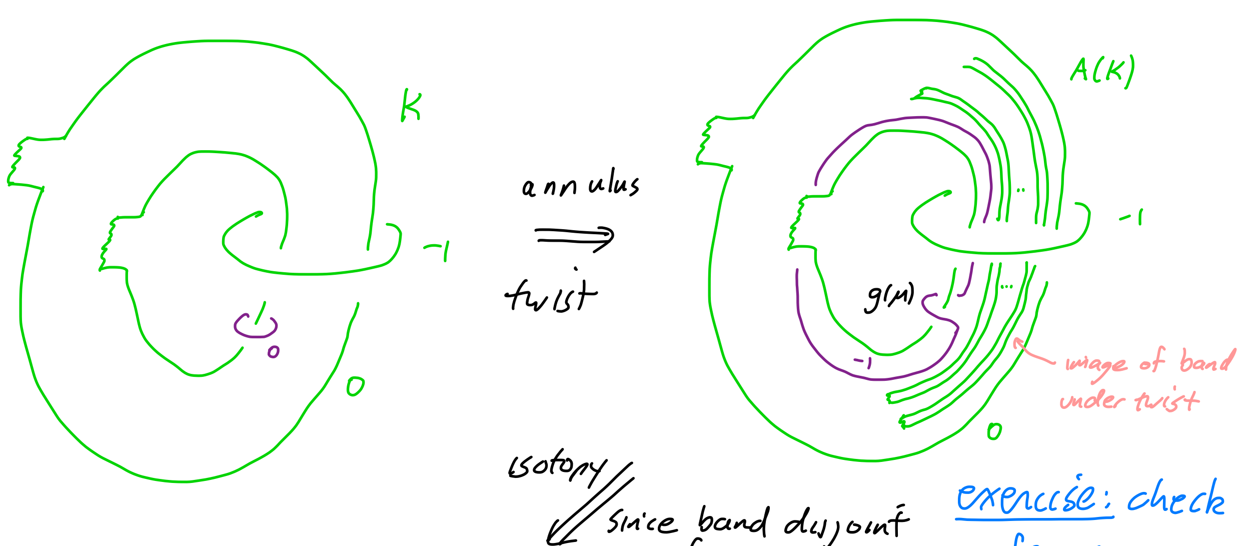
thus G extends over B^4 to give a diffeom. from $X_K(n)$ to $X_{K'}(n)$ $\blacksquare$

Proof of Thm 4:

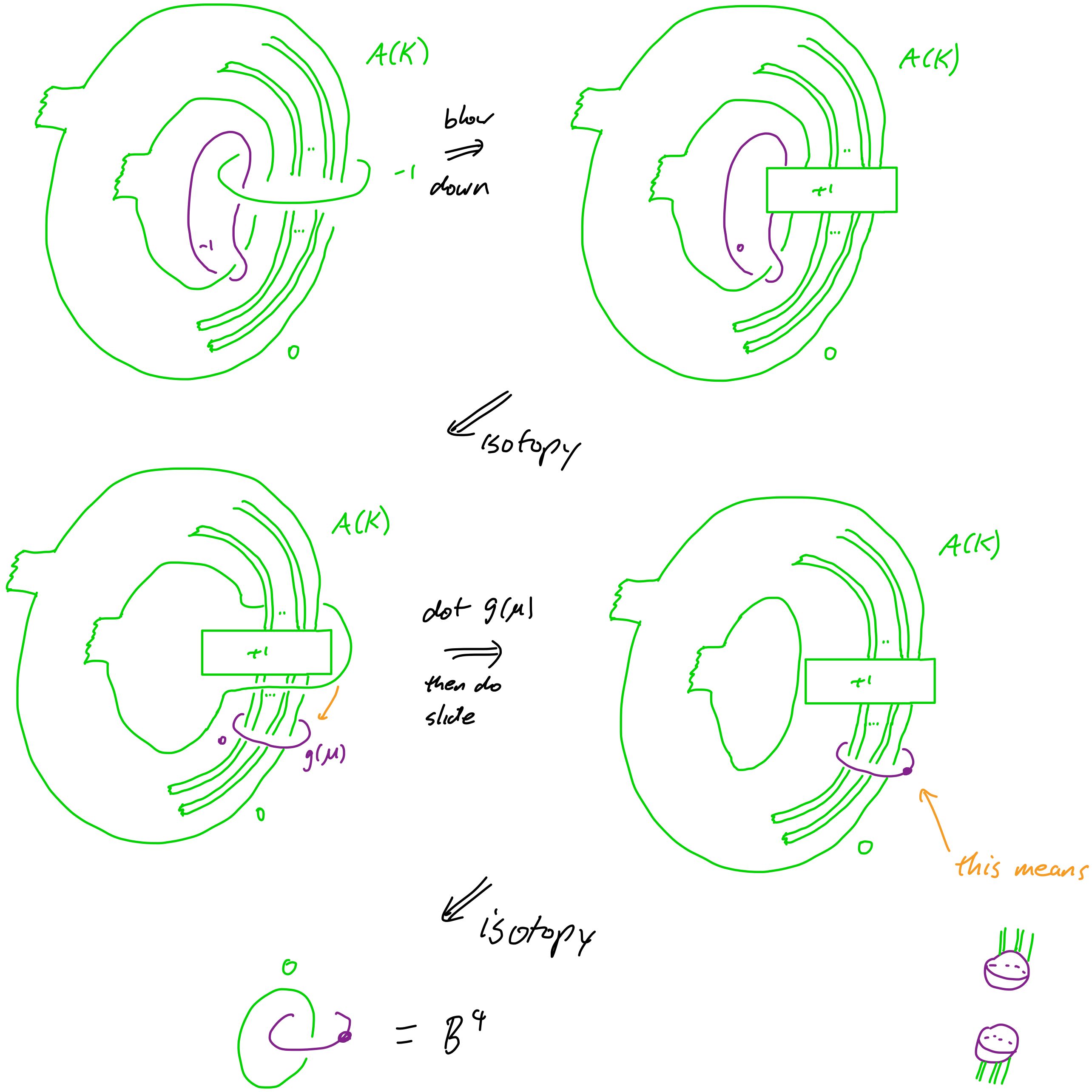
Since K has a special annulus presentation it looks like



Now we have the meridian μ to K



exercise: check framing



so we can apply lemma 5 to see $X_K(0) \cong X_{A(K)}(0)$

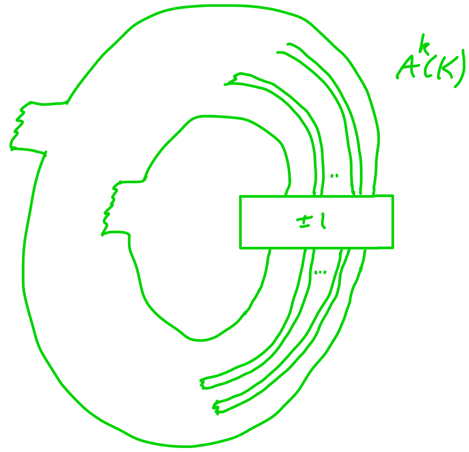
now iterate to get $X_K(0) \cong X_{A^n(K)}(0)$

<u>exercise:</u> Modify above if annulus was twisting -1

What about for $n \neq 0$?

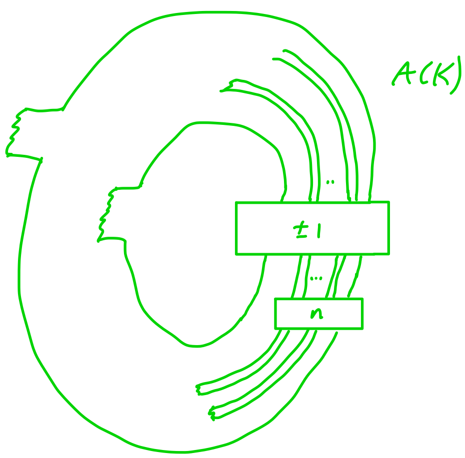
let K have a special annulus presentation

we can write $A^k(K)$ as



(number of bands in box depends on k)

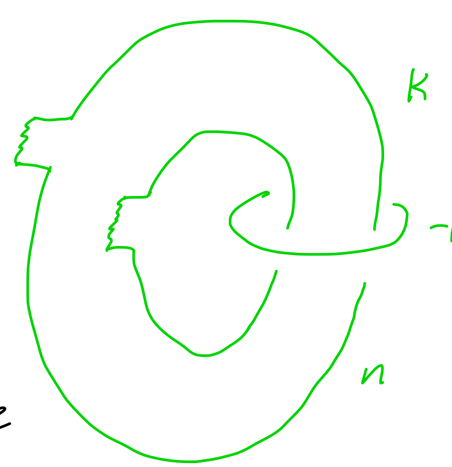
now denote by $A^k_n(K)$ the knot



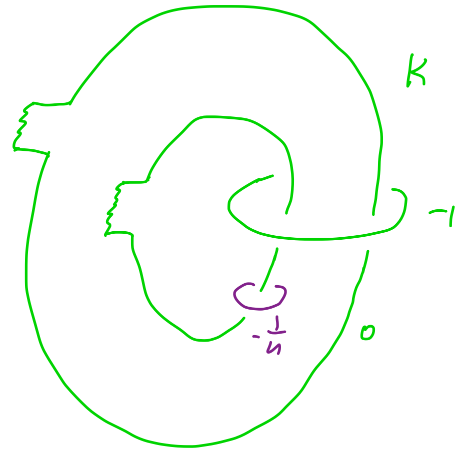
Theorem 6 (Abe, Jong, Lueke, Osoinach 2015):

> for any n and all k,
>
> $$X_K(n) \cong X_{A^k_n(K)}(n)$$
>
> in particular $S^3_K(n) \cong S^3_{A^k_n(K)}(n)$

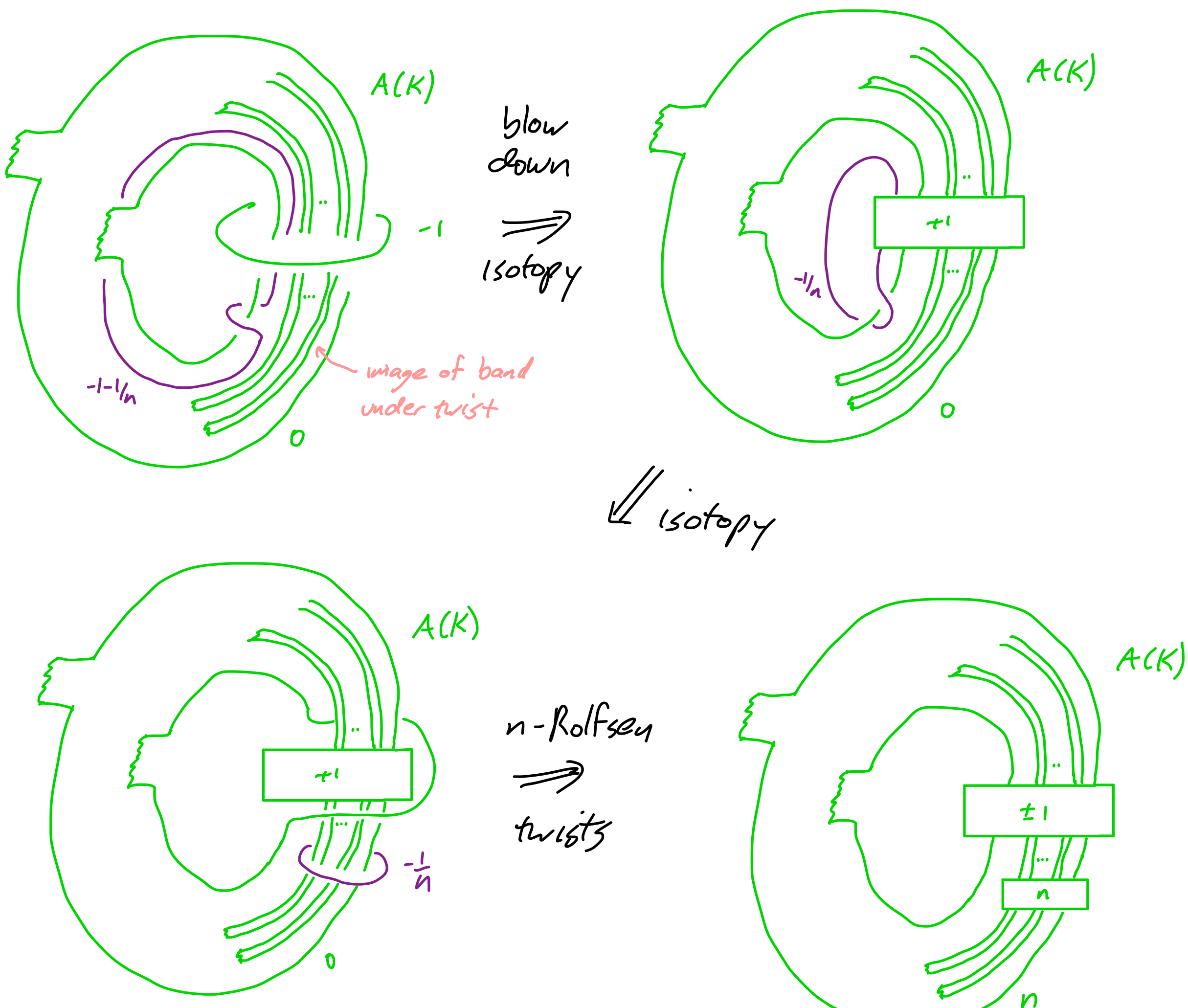
Proof: we consider the case



and begin by showing the 3-mfds are the same we rewrite the above as



performing an annulus twist on this picture gives a diffeom manifold given by



so $S^3_K(n) \cong S^3_{A'_n(K)}(n)$

can iterate construction to get result for all k

the proof that $X_K(n) \cong X_{A_n^k(K)}(n)$ is now exactly as in the proof of Th$^{\underline{m}}$ 4 ▦

Cor 7: $\boxed{\text{If } K \text{ is as in example above, then } A_n^k(K) \text{ different for each } k, \text{ so } \exists\, \infty\text{'ly many knots in } S^3 \text{ on which } n\text{-surgery yields the same 3-manifold and have the same } n\text{-traces}}$

Proof: for $n=0$, this is in corollary 3

for $n \neq 0$ Abe, Jong, Lueke, Osoinach show that

$$\deg \Delta_{A_n^{k+1}(K)}(t) > \deg \Delta_{A_n^k(K)}(t)$$

Alexander polynomial.

we skip the proof as it is a bit far afield ▦

B Dualizable Patterns

a pattern is an embedding $P: S^1 \to V$ where $V = S^1 \times D^2$

(we assume im $P \not\cong S^1 \times \{pt\}$)

given a knot K in S^3 and a framing $\mathcal{F}$ on K

$\exists$ an embedding $\iota_{\mathcal{F}}: V \to S^3$ such that

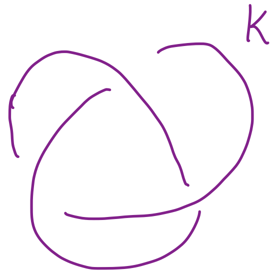
$\iota_{\mathcal{F}}(V) =$ nbhd of K and

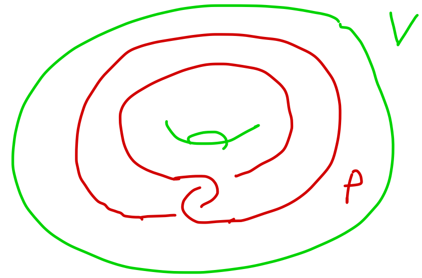
$\iota_{\mathcal{F}}(S^1 \times \{p\})$ defines $\mathcal{F}$ for any $p \in \partial D^2$

the satellite of K by P is the knot $\iota_{\mathcal{F}} \circ P: S^1 \to S^3$

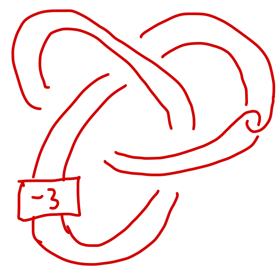
and denoted $P_{\mathcal{F}}(K)$ (if $\mathcal{F}=0$, then just $P(K)$)

example:





$P(K)$ is



a pattern $P: S^1 \to V = S^1 \times D^2$ is called <u>dualizable</u> if $P(S^1)$ is not null-homologous and $\exists$ a pattern $P^*: S^1 \to V^* = S^1 \times D^2$ such that $\exists$ an orientation preserving diffeomorphism

$$f: [V - \underbrace{N(P(S^1))}_{\text{nbhd } P(S^1)}] \longrightarrow [V^* - \underbrace{N(P^*(S^1))}_{\text{nbhd } P^*(S^1)}]$$

with $f(\lambda_V) \underset{\text{isotopic}}{\simeq} \lambda_{P^*}$, $\underline{f(\lambda_P) \simeq \lambda_{V^*}}$, $\underline{f(\mu_V) \simeq -\mu_{P^*}}$

where $\lambda_V = S^1 \times \{p\} \quad p \in \partial D^2$

λ_P = unique curve on $\partial N(P(S^1))$ homologous to a positive multiple of λ_V in $V - N(P(S^1))$

$\mu_V = \{q\} \times \partial D^2 \quad$ any $q \in S^1$

μ_P = meridian to $P(S^1)$ on $\partial N(P)$

and similarly for $\lambda_{V^*}, \mu_{V^*}, \mu_{P^*}$

exercise: Show if $\exists$ an $f: [V \setminus N(P(S^1))] \longrightarrow \overbrace{[V^* \setminus N(P^*(S^1))]}^{V^* - N(P^*)}$ such that $f(\lambda_P) \simeq \lambda_{V^*}$ and $f(\mu_V) \simeq -\mu_{P^*}$

then can isotop f so that $f(\lambda_V) = \lambda_{P^*}$ and $f(\mu_P) = -\mu_{V^*}$

What are dualizable patterns good for?

Th$^{\underline{m}}$ 8 (Brakes 1980):

> If P is a dualizable pattern with dual P^*, then there is a diffeomorphism $\phi: S^3_{P(U)}(0) \to S^3_{P^*(U)}(0)$ where U is the unknot

Proof: let $V_P = V - N(P(S^1))$ $\quad \partial V_P = T_1 \cup T_2$ ← ∂V

and $V^*_{P^*} = V^* - N(P^*(S^1))$ $\quad \partial V^*_{P^*} = T_1^* \cup T_2^*$ ← ∂V^*

note: $V(\lambda_V) \cong S^3 \cong V^*(\lambda_{V^*})$

now $S^3_0(P(U)) = V_P(\lambda_P, \lambda_V)$

↑ Dehn fill T_1 by w/ slope λ_P and T_2 by slope λ_V

indeed note that since λ_P is homologous to some multiple of λ_V in V_P, $\exists$ a surface $\Sigma' \subset V_P$ s.t. $\partial \Sigma' = \lambda_P \cup n\lambda_V$ ← n copies

so $\Sigma = \Sigma' \cup n$ meridional disks in the filling torus $S^1 \times D^2$ for T_2 is a Seifert surface for $P(U)$

that is λ_P is the 0 framing on $P(U)$

similarly $S^3_0(P^*(U)) = V^*_{P^*}(\lambda_P, \lambda_V)$

and we have the diffeomorphism

$$\begin{array}{ccccccc} S^3_0(P(U)) = & V_P & \cup_{T_1} & S^1 \times D^2 & \cup_{T_2} & S^1 \times D^2 \\ & \downarrow f & & \downarrow id & & \downarrow id \\ S^3_0(P^*(U)) = & V^*_{P^*} & \cup_{T_1^*} & S^1 \times D^2 & \cup_{T_2^*} & S^1 \times D^2 \end{array}$$

exercise: Use lemma 5 to show $X_{P(U)}(0) \cong X_{P^*(U)}(0)$

let $\tau_n : S^1 \times D^2 \to S^1 \times D^2 : (\phi, (r, \theta)) \longmapsto (\phi, (r, \theta + n\phi))$

define $\tau_n(P) = \tau_n \circ P$, this is a new pattern in V

Th^m 9 (Miller-Piccirillo 2018):

> let P be a dualizable pattern with dual P^*, then for any $n \in \mathbb{Z}$
> $$S^3_{P(U)}(n) \cong S^3_{(\tau_n(P))(U)}(n)$$

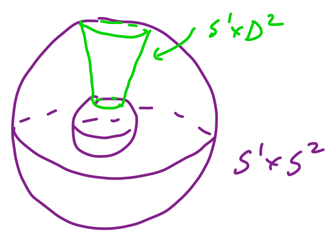
Proof: exercise. very similar to proof of Th^m 8 $\boxplus$

OK great, but do dualizable patterns exist?

to find them we set

$$\Gamma : S^1 \times D^2 \to S^1 \times S^2 : (x,y) \mapsto (x, e(y))$$

where $e : D^2 \to S^2$ maps D^2 to a nbhd of north pole



if $\alpha : S^1 \to S^1 \times D^2$ then let $\hat{\alpha} = \Gamma \circ \alpha$

Th^m 10 (Miller-Piccirillo 2018):

> a pattern P in $S^1 \times D^2$ is dualizable $\Leftrightarrow$ $\hat{P}$ is isotopic to $\hat{\lambda}_V$ in $S^1 \times S^2$

Proof: ($\Rightarrow$) note $S^1 \times S^2 \setminus N(\hat{P})$ is diffeomorphic to $(S^1 \times D^2 \setminus N(P))_{T_2}(\mu_V)$ Dehn fill $\partial S^1 \times D^2$ along μ_V

since P is dualizable with dual P^*, $\exists$ a diffeo. $f : (V \setminus N(P)) \to (V^* \setminus N(P^*))$

sending μ_V to $-\mu_{P^*}$

so $(S^1 \times D^2 \setminus N(P))_{T_2}(\mu_V)$ is diffeomorphic to $(S^1 \times D^2 \setminus N(P^*))_{T_1}(-\mu_{P^*})$

but this is just a solid torus

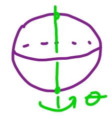
so $\hat{P}$ is a knot in $S^1 \times S^2$ with solid torus complement.

i.e. $\partial N(\hat{P})$ is a Heegaard torus for $S^1 \times S^2$

it is known (Waldhausen 1968) that $S^1 \times S^2$ has a unique Heegaard torus so $\partial N(\hat{P})$ is isotopic to a nbhd of $\hat{\lambda}_V$ and thus $\hat{P}$ is isotopic to $\hat{\lambda}_P$

($\Leftarrow$) let $V^* = S^1 \times S^2 \setminus N(\hat{P})$

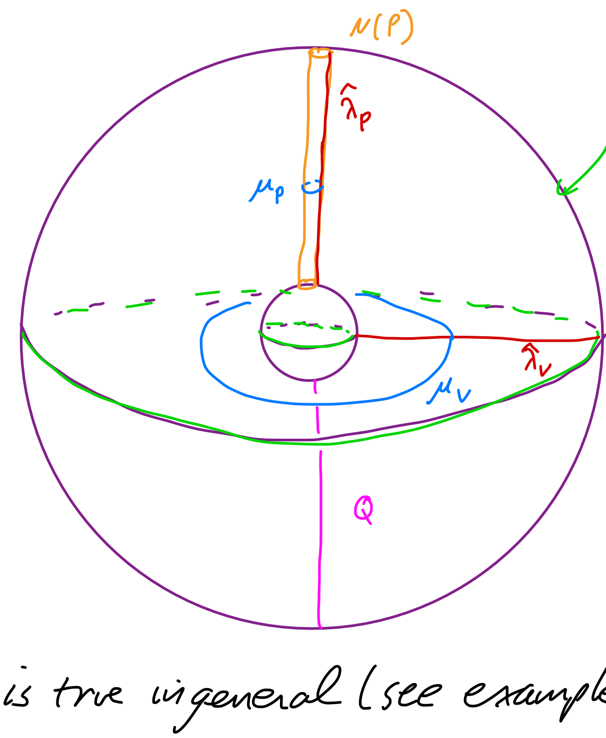
Since $\hat{P}$ is isotopic to $\hat{\lambda}_V \simeq S^1 \times \{pt\}$ we know that V^* is a solid torus so $\exists$ a diffeomorphism of $f: V^* \to S^1 \times D^2$ such that $f(\hat{\lambda}_P) \simeq S^1 \times \{pt\} = \lambda_{V^*}$

note: $T: S^1 \times S^2 \to S^1 \times S^2 : (\theta, x) \mapsto (\theta, r_\theta(x))$, where $r_\theta : S^2 \to S^2$ rotates S^2 by θ , changes framing on $\hat{\lambda}_P$

let $Q = \hat{\lambda}_V \subset V^*$ and $Z = (S^1 \times S^2) \setminus N(\hat{P} \cup \hat{\lambda}_P)$

note: $V \setminus N(P) \cong Z \cong V^* \setminus N(\hat{\lambda}_V)$

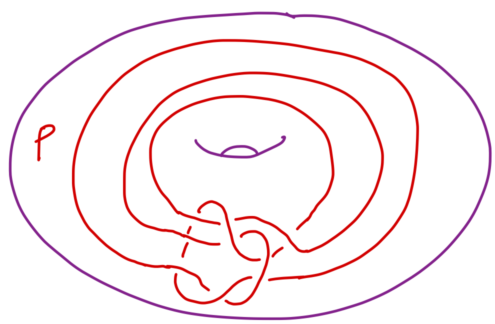
in the "trivial case" we see $\mu_V \leftrightarrow -\mu_Q$ and $\lambda_P \leftrightarrow \lambda_{V^*}$ in these diffeomorphisms



this is true in general (see example below)

so P is dualizable with $P^* = f(Q) \subset S^1 \times D^2$

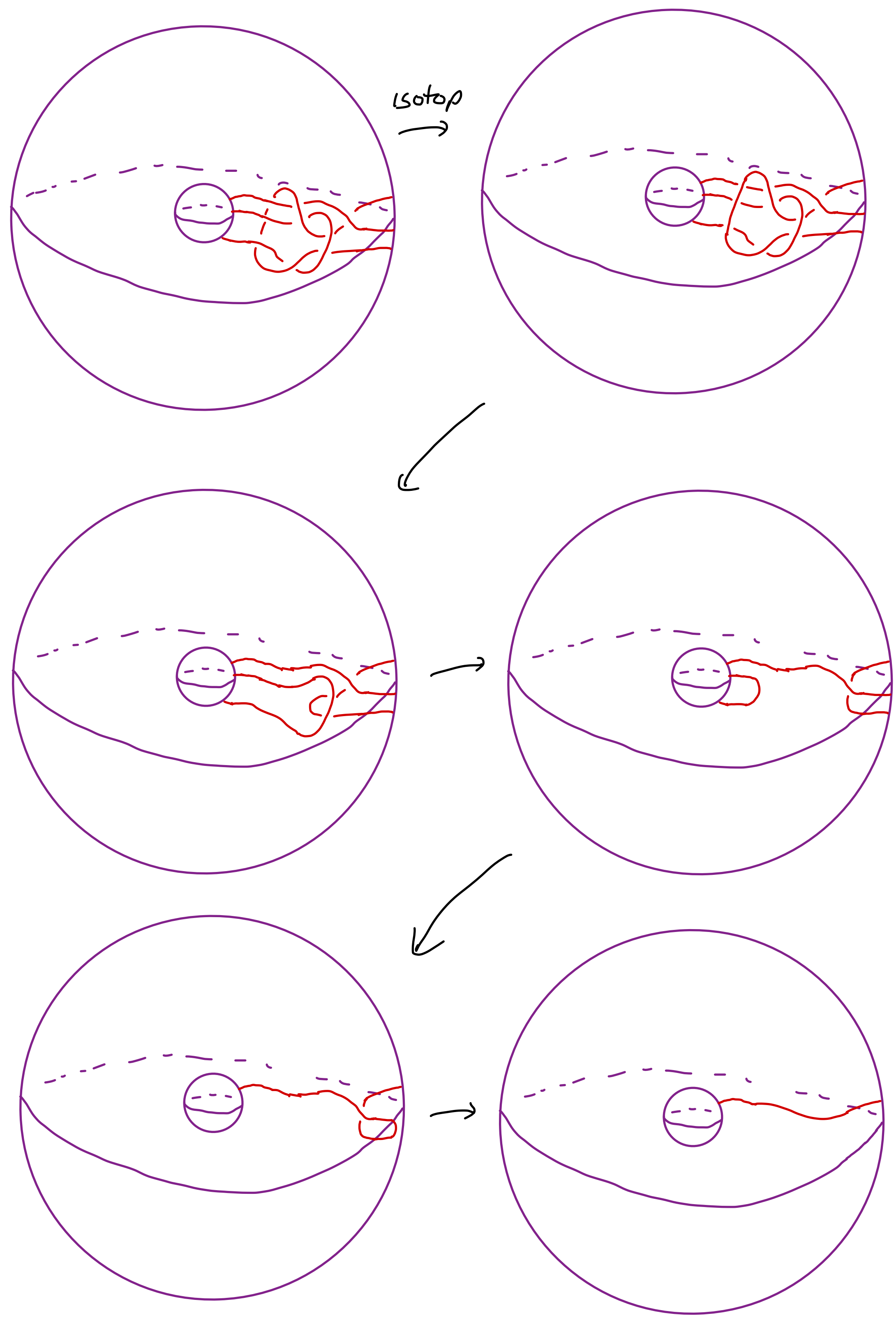
example:



is dualizable

indeed

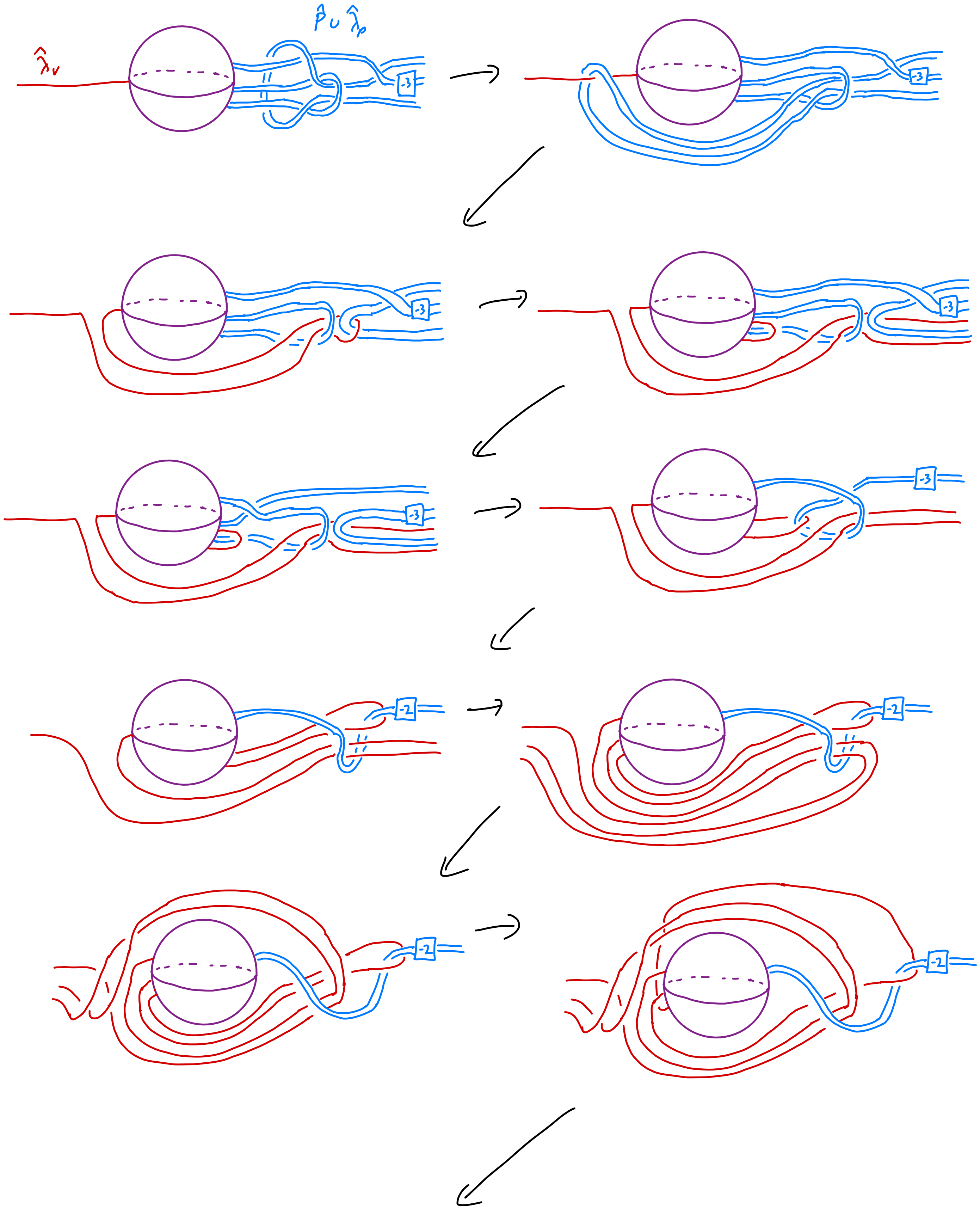
$\hat{P}$ is

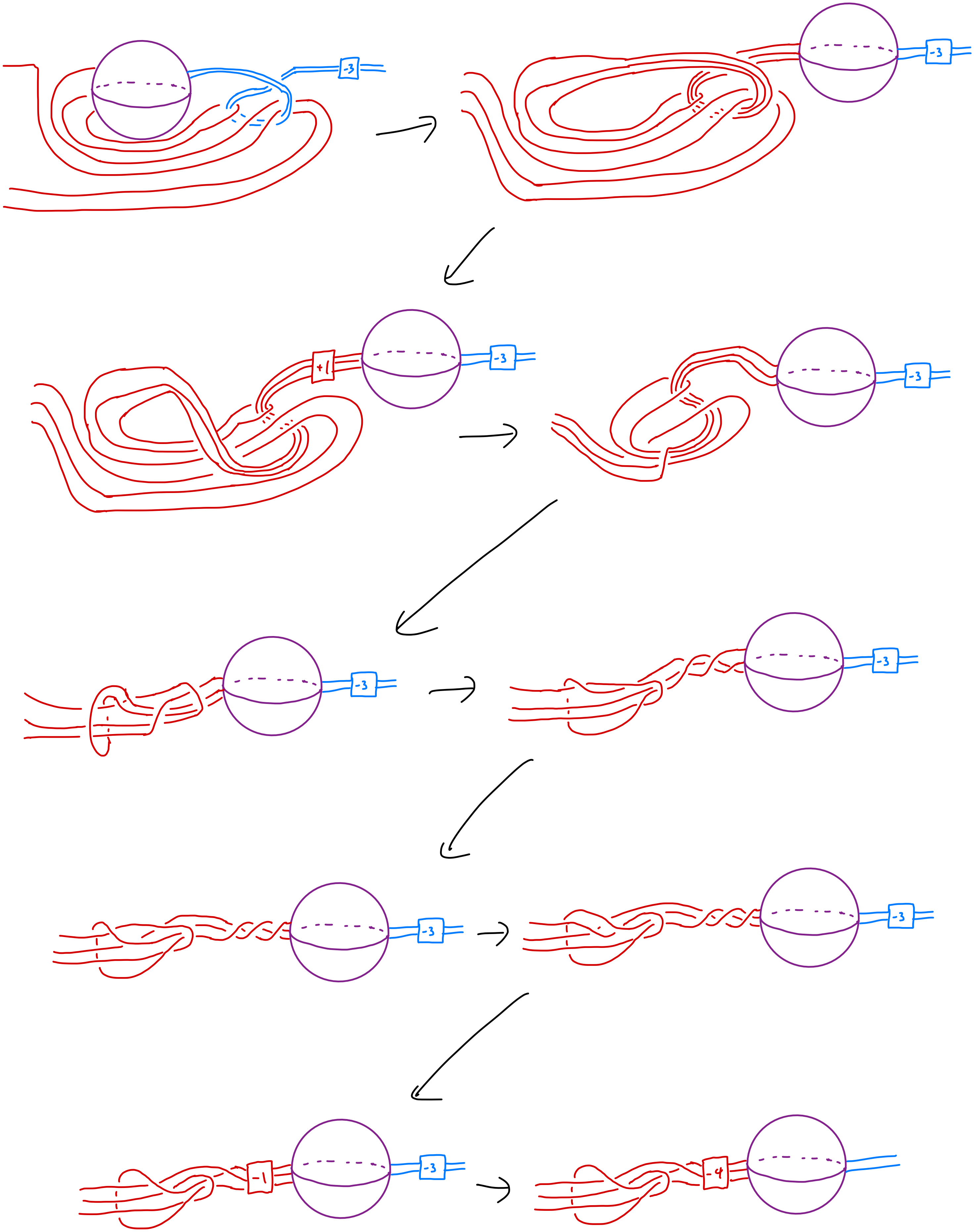


We now see the dual of P is $\tau_{-4}(P)$

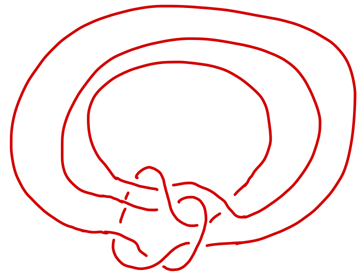
to do this we draw $\hat{\lambda}_v$ and $\hat{P}$ together with $\hat{\lambda}_p$ (the framing on P)

(dropping outer S^2 from the picture)

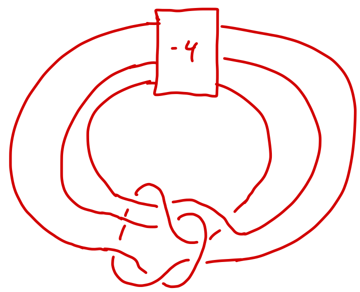



-3
+1
-1
-4

so 0-surgery on



and on



are diffeomorphic!

exercise:

if P is dualizable with dual P^*, then $\tau_n(P)$ is dualizable with dual $\tau_{-n}(P^*)$

(so for P the pattern above $\tau_n(P)$ has dual $\tau_{-4-n}(P)$)

2 knots K_0, K_1 are called concordant if there is an embedded annulus $A \subset S^3 \times [0,1]$ s.t. $A \cap S^3 \times \{i\} = K_i$ $i = 0,1$

Akbulut-Kirby conjectured that if $S^3_K(0) \cong S^3_{K'}(0)$ then K and K' are concordant

Th$\underline{\text{m}}$ 11 (Miller-Piccirillo 2018):

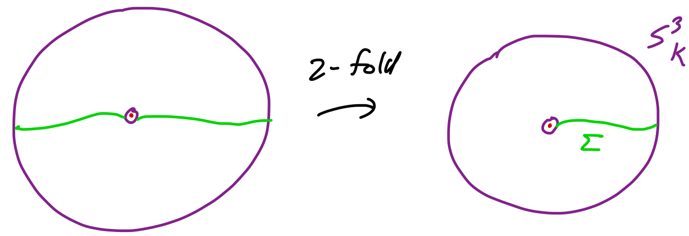
$\exists$ infinitely many pairs K, K' that are not concordant but $S^3_K(0) \cong S^3_{K'}(0)$

Proof: given a knot $K \subset S^3$

let $\Sigma_2(K)$ be the 2-fold cover of S^3 branched over K

that is consider the 2-fold cover of S^3_K corresponding to the subgroup $\ker(\pi_1(S^3_K) \to H_1(S^3_K) \to \mathbb{Z}/2)$

then glue in a solid torus so that its meridian goes to the lift to the meridian of K



exercise: If K, K' are concordant, then $\exists$ a compact 4-manifold X s.t. $\partial X = -\Sigma_2(K) \cup \Sigma_2(K')$ and $H_*(X, -\Sigma_2(K)) \cong H_*(X, \Sigma_2(K')) \cong 0$ (X called homology cobordism)

let $K_n = (\tau_{2n-1} J)(U)$ and $K_n' = (\tau_{-3-2n} J)(U)$

where J is as above

from Thm 8 we know $S^3_{K_n}(0) \cong S^3_{K_n'}(0)$

to show K_n is not concordant to K_n' Miller and Piccirillo compute Ozsváth and Szabó's d-invariants

one can show $H_*(\Sigma_2(K_n)) \cong H_*(\Sigma_2(K_n')) \cong H_*(S^3)$

so the d-invariant of $\Sigma_2(K_n)$ and $\Sigma_2(K_n')$ is a rational number and it is known that is two homology spheres are homology cobordant then their d-invariants are the same

Miller-Piccirillo computed $d(\Sigma_2(K_n')) \le -2 < 0 \le d(\Sigma_2(K_n))$

see their paper

If $Q: S^1 \to V$ is a pattern, then let $j_Q: S^1 \times D^2 \to V$ parameterize a nbhd $N(Q(S^1))$ such that $j_Q(S^1 \times \{p\}) \simeq \lambda_Q$

given another pattern $P: S^1 \to V \cong S^1 \times D^2$ define the composition

$$P \circ Q = j_Q \circ P$$

Th^m 12:

$$\boxed{\text{If } P \text{ and } Q \text{ are dualizable, with duals } P^* \text{ and } Q^*; \text{ then } P \circ Q \text{ is dualizable with dual } Q^* \circ P^*}$$

Proof: we denote the solid torus in which a pattern R lives by V_R

note: $V_{P\circ Q} \setminus N(P\circ Q) = (V_Q \setminus N(Q)) \cup_\sim (V_P \setminus \nu(P))$

where λ_{V_P} is identified with λ_Q
and μ_{V_P} " " μ_Q

since P and Q are dualizable, we see

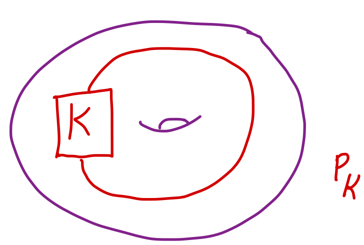
$$V_{Q\circ P} \setminus N(Q\circ P) \cong (V_{P^*} \setminus N(P^*)) \cup_\sim (V_{Q^*} \setminus N(Q^*))$$

where λ_{P^*} is identified with $\lambda_{V_Q^*}$
and μ_{P^*} with $-\mu_{V_Q^*}$

but this is exactly $V_{Q^*\circ P^*} \setminus N(Q^* \circ P^*)$

exercise: check diffeo sends $\lambda_{V_{P\circ Q}}$ to $\lambda_{Q^*\circ P^*}$ and $\mu_{P\circ Q}$ to $-\mu_{V_{Q^*\circ P^*}}$ ∎

exercise: given $K \subset S^3$ we can get a pattern P_K



1) show $P_K(K') = K \# K'$

2) show P_K is dualizable with dual P_K

Cor 13:

$\boxed{\text{If } P \text{ is a dualizable pattern and } K \text{ a knot in } S^3, \text{ then } S^3_{P(K)}(0) \cong S^3_{P^*(U)\#K}(0)}$

Proof: Since $P_K(U) = K$ we see $P \circ P_K(U) = P(K)$

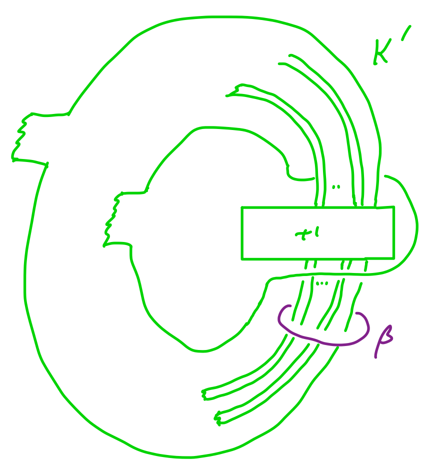
now $(P \circ P_K)^* = P_K \circ P^*$ so $P_K \circ P^*(U) = K \# P^*(U)$

the result follows from Th$^{\underline{m}}$ 8 ▨

Th$^{\underline{m}}$ 14 (Miller-Piccirillo 2018)

let K admit a special annulus presentation, and K' be obtained by an annulus twist
Then there is a dualizable pattern P such that $P(U) \cong K'$ and $P^*(U) \cong K$

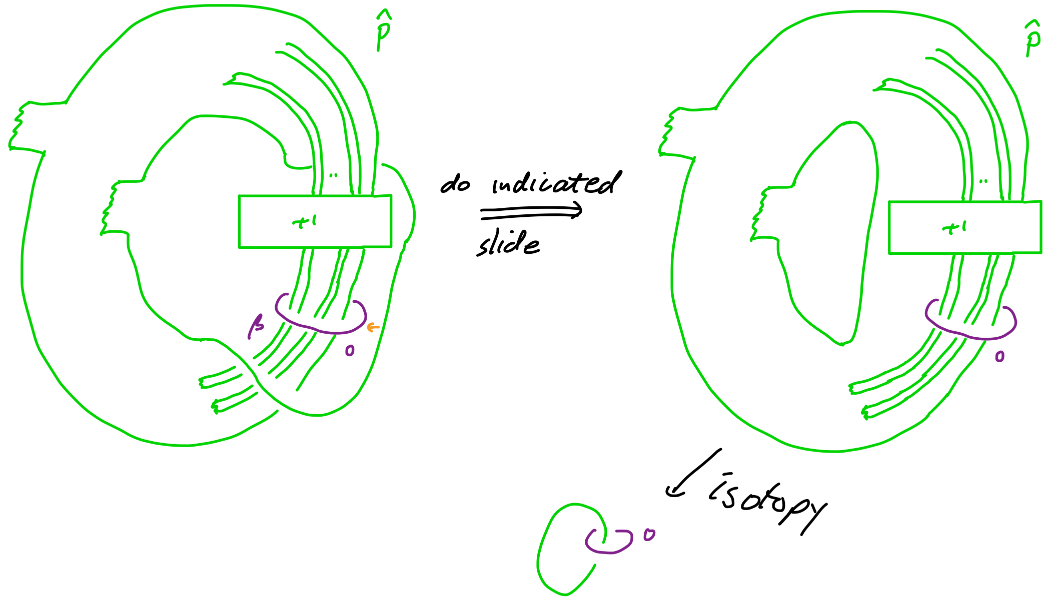
Proof: recall K' looks like



let $V = \overline{S^3 - nbhd(\beta)}$ and $P = K' \subset V$

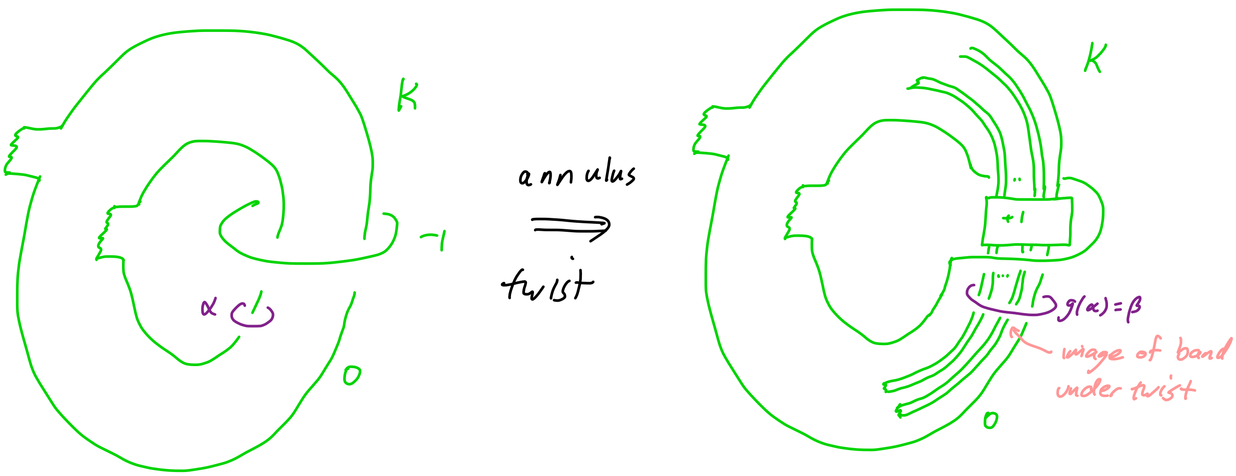
to see P is dualizable we use Th$^{\underline{m}}$ 10 and see $\hat{P} \subset S^1 \times S^2$ is isotopic to $S^1 \times \{pt\}$

$\hat{P} \subset S^1 \times S^2$ is shown in the figure above if we do zero surgery on β



so P is dualizable with some dual P^*

now to see what P^* is consider the homeo $g: S^3_K(0) \to S^3_{K'}(0)$



so $S^3 \setminus N(K) \cong S^3_K(0) \setminus N(\alpha) \underset{g}{\cong} S^3_{K'}(0) \setminus N(\beta)$

now $S^3_{K'}(0) \setminus N(\beta)$ is the result of filling $(V \setminus N(P))$ along λ_P

which (by def^n) is homeomorphic $(V^* \setminus N(P^*))_{\tau_2}(\lambda_{V^*})$

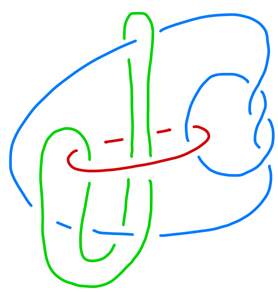
which in turn is $S^3 \setminus N(P^*(U))$

but Gordon-Luecke showed a knot is determined by its complement, so K isotopic to $P^*(U)$

C. RGB Links

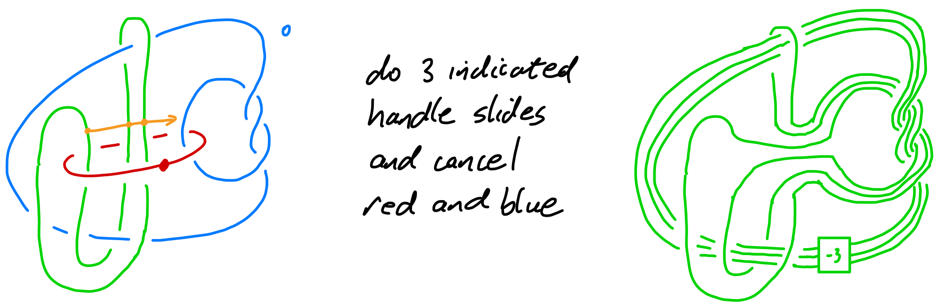
an RGB link is a 3-component link whose components are written R, G, B such that
1) $B \cup R$ is isotopic to $B \cup \mu_B$ ← meridian to B
2) $G \cup R$ is isotopic to $G \cup \mu_G$ ← meridian to G
3) $lk(B, G) = 0$

example:

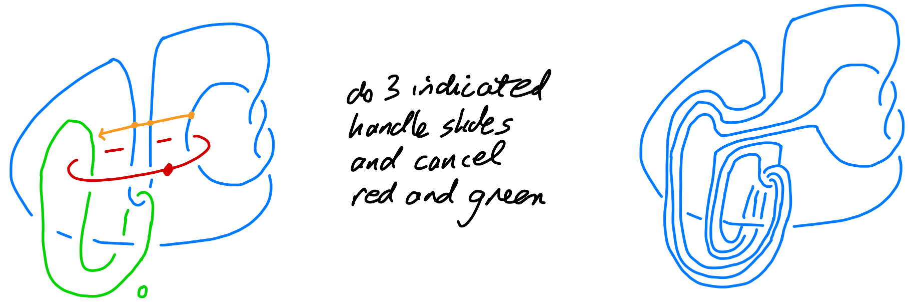


by property 1) if we attach a 1-handle to B^4 by putting a dot on R and attach a 0-framed 2-handle to B we get B^4

the link G becomes a knot K_G in S^3



by property 2) we also get K_B in S^3



by property 3) the 0-framing on B and G goes to the 0-framing on K_B and K_G

this proves

Th$^{\underline{m}}$ 15 (Piccirillo 2019):

$$\boxed{X_{K_G}(0) \cong X_{K_B}(0)}$$

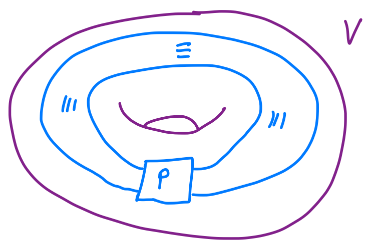
exercise: Show if you attatch 2-handles to G, B with framing 0 and n respectively, and dot red, then you get K_G^n and K_B^n s.t.

$$X_{K_G^n}(n) \cong X_{K_B^n}(n)$$

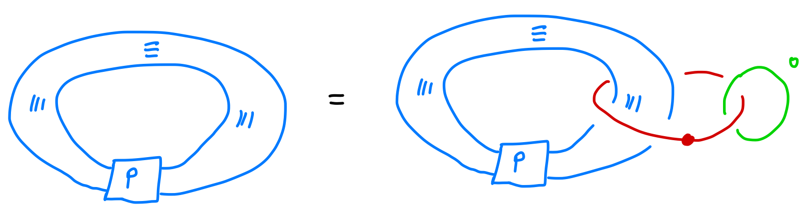
Th$^{\underline{m}}$ 16 (Piccirillo 2019):

> If P is a dualizable pattern and P^* is its dual then $\exists$ an RGB link s.t. $P(U) \simeq K_B$ and $P^*(U) \simeq K_G$

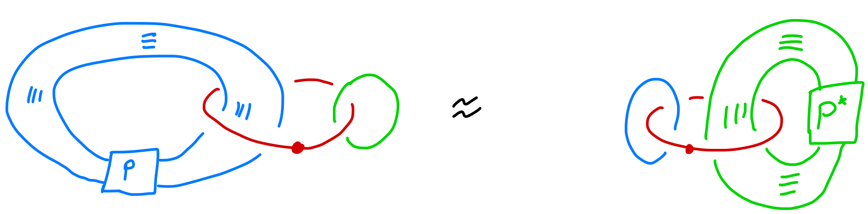
Proof: suppose P is



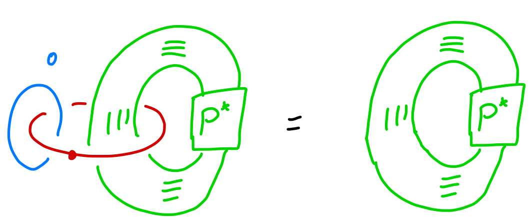
now P(U) is



but $\partial(\bigcirc) = S^1 \times S^2$ so by Th$^{\underline{m}}$ 10



and $P^*(U)$ is

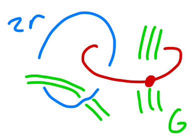


Thm 17 (Piccirillo 2019):

$$\boxed{\text{Given an RGB link, then } \exists \text{ a dualizable pattern } P \text{ with dual } P^* \text{ such that } K_G \simeq P(U) \text{ and } K_B \simeq P^*(U)}$$

Proof: Given RBG link, put a dot on R and slide B "over" R until it is an unknot (recall $\partial(0^\circ) \cong \partial(\odot)$)

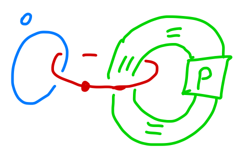
suppose this took r slides (counted with sign) so framings on B change by 2r



slide B over R (-r) times to get change in framing to be 0

note B and R bound disks if G intersects B's disk

slide it "over" red to get



now G is in the form of P(U)

one can do the same to get $B = P^*(U)$ ▦